ABSTRACT

Consider a city with a highly compact central business district (CBD), and in which commuters’ origins are continuously dispersed. The travel demand is dependent on the total travel cost to the CBD. The transportation system is divided into two layers: major freeways and dense surface streets. Whereas the major freeway network is modeled according to the conventional discrete network approach, the dense surface streets are approximated as a continuum. Travelers to the CBD either travel in the continuum (surface streets) and then exchange to the discrete network (freeways) at an interchange (ramp) before moving to the CBD on the discrete network, or travel directly to the CBD in the continuum. Specific travel cost-flow relationships for the two layers of transportation facilities are considered. We develop a traffic equilibrium model for this discrete/continuous transportation system, in which for a particular origin no traveler can reduce their individual travel cost to the CBD by unilaterally changing routes. The problem is formulated as a simultaneous optimization program with two sub-problems. One sub-problem is a traffic assignment problem from the interchanges to the CBD in the discrete network, and the other is a traffic assignment problem with multiple centers (i.e. the interchange points and the CBD) in the continuous system. A Newtonian algorithm that is based on the sensitivity analyses of the two sub-problems is proposed to solve the resultant simultaneous optimization program. A numerical example is given to demonstrate the effectiveness of the proposed methodology.
INTRODUCTION

Due to the rapid development of the traffic equilibrium problem during recent decades, a large literature is now devoted to the theory of equilibrium prediction and to computer algorithms for the determination of flows. Two general approaches are used to deal with traffic equilibrium problems. The first is the discrete modeling of the transportation network, in which the zones are identified as nodes in space and the roads are treated as links between these nodes. The other approach is the continuous modeling of the transportation systems. The major assumption of this form of modeling is that differences between adjacent areas within a city are relatively small as compared to the variation over the entire city, and hence the characteristics of the street system can be represented by smooth mathematical functions (1). A number of studies on the continuous modeling of transportation systems can be found in the literature, and they can be broadly classified into two categories: specific city configuration approaches (2-5) and general city configuration approaches (6-11).

Recently, Taguchi and Iri (12) proposed a promising numerical procedure to solve the problem of continuous transportation system for a general city configuration, in which the finite element method (13) was employed to solve three continuum problems: the maximum flow problem, the shortest route problem, and the minimum-cost flow problem. For user equilibrium problems, a dual-based formulation was given by Sasaki et al. (14), in which the user equilibrium problem in the continuous system was solved by minimizing an objective function that was subject to a set of constraints. The finite element method was also used to determine the cost potential in the city, and the flow intensity was then deduced from the potential function. To improve the numerical stability of the solution to the problem, a more robust algorithm was developed based on the mixed finite element formulation (15), which was also applied to solve the continuous modeling of a multi-commodity traffic assignment with fixed and variable demands (16), and a combined distribution and assignment problem (17, 18). For the problems of competitive facilities, Wong and Yang (19) developed a continuum formulation for the determination of the market areas of competitive facilities, and then significantly extended it to include incorporation of the demand elasticity of customers over the space and the market externality of each competitive facility (20). This extension added realism because congestion externalities and demand elasticity do exist in practice.

Although the traffic equilibrium problem has been vigorously formulated and analyzed in considerable depth for both discrete and continuous approaches, both approaches have their own advantages and shortcomings, and it is promising to integrate these two approaches to form a combined model. One of the earliest attempts at discrete/continuous modeling was made by D’Este (4), who considered the problem of flow-dependent trip assignment for a city with a small number of radial major roads. A system of differential equations was derived for the spatial pattern of trip assignment in a model city with a continuous distribution of home locations and a ring-radial road network. Wong (5) reformulated the problem as a minimization problem of an objective function that was subject to a set of constraints and solved by means of power series expansion. For a general city configuration, Yang et al. (21) developed a dual-based formulation and a finite element solution for a discrete/continuous network problem. In their approach, the solution algorithm is applicable to a special case in which there is only one feasible path from each interchange to the CBD. This restricts the solution’s applicability to a more general discrete/continuous transportation system configuration. In this paper, we reformulate the discrete/continuous transportation problem for the general city configuration and propose an efficient solution algorithm to solve the resultant problem with multiple feasible routes in the discrete network.
We consider a city with a highly compact central business district (CBD). In the city, the commuters’ origins are continuously dispersed. We assume that the travel demand is dependent of the total travel cost to the CBD. The transportation system is divided into two layers: major freeways and dense surface streets. Whereas the major freeway network is modeled with the conventional discrete network approach, the dense surface streets are approximated as a continuum. There are two possibilities for the travelers to move to the CBD. They can either travel in the continuum (surface streets) and exchange to the discrete network (freeways) at an interchange (ramp) before moving to the CBD on the discrete network, or travel directly to the CBD in the continuum. Specific travel cost-flow relationships for the two layers of transportation facilities are considered. We develop a traffic equilibrium model for the above discrete/continuous transportation system, in which for a particular origin no traveler who can reduce their individual travel cost to the CBD by unilaterally changing routes. This approach provides a more realistic model for the transportation system, in which the travel demand is continuously distributed over the city rather than concentrated at some arbitrarily chosen point sources (centroids), and most travelers tend to find a nearby interchange to move to the discrete freeway network to take advantage of high speed movements, except those who are close to the CBD. The problem is formulated as a simultaneous optimization program that consists of two sub-problems. One sub-problem is a traffic assignment problem from the interchanges to the CBD in the discrete network, and the other is a traffic assignment problem with multiple centers (i.e. the interchange points and the CBD) in the continuous system. A Newtonian algorithm, which is based on the sensitivity analyses of the two sub-problems, is proposed to solve the resultant simultaneous optimization program that is shown to belong to the class of fixed point problems.

This paper is organized as follows. We first introduce the modeled city with our assumptions, definitions, and notation, and then describe the problem formulation. A solution algorithm is proposed to solve the resultant problem, and a numerical example is given to demonstrate the effectiveness of the proposed methodology.

MODELED CITY

Consider a city with a CBD as shown in Figure 1a, in which the transportation system consists of two sub-systems: a discrete system for the freeway links and a continuous system for the dense surface network. The discrete and continuous systems interact at the interchange locations (ramps) at which the travelers are collected to board the freeway network, and the CBD location at which the freeway users enter the CBD.

Continuous System

The dense surface road network is approximated as a continuum, as shown in Figure 1c (14). There are $N$ interchanges. The CBD and all of these interchanges are sufficiently compact as compared to the continuous domain of the city. The travelers to the CBD either travel in the continuum (surface streets) and then exchange to the discrete network (freeways) at an interchange (ramp) before moving to the CBD on the discrete network, or travel directly to the CBD in the continuum. Denote the continuous region of the city as $\Omega$, the boundary of the city as $\Gamma$, the location of the CBD as $O_0$, and the location of the interchange $n$ as $O_n$. The CBD or each of the interchanges is embraced by a close boundary, $\Gamma_n$, $n = 0, 1, 2, \ldots, N$. The distribution of the travelers over the city $\Omega$ is assumed to be continuous and represented by a
non-negative, heterogeneous density function \(q(x, y)\) where \(q\) is the total demand per unit area from the home location \((x, y) \in \Omega\). To consider the elasticity of travel demand, \(q(x, y)\) is assumed to be a function of the minimum travel cost

\[
q(x, y) = D(u(x, y), x, y),
\]

(1)

where \(u(x, y)\) and \(q(x, y)\) are the minimum travel cost and the travel demand that are generated from the location \((x,y) \in \Omega\) to travel to the CBD. The function \(D(.)\) is assumed to be monotonically decreasing to reflect the elastic nature of travel demand with respect to the total cost, and its inverse function exists.

The local travel cost in the continuous domain of city is assumed to be dependent on the local flow intensity and road configuration, but not upon direction (the isotopic case),

\[
c(x, y, f) = \alpha(x, y) + \beta(x, y)|f(x, y)|^{\gamma(x, y)},
\]

(2)

where \(c(x, y, f)\) is the cost per unit distance of travel at co-ordinate \((x,y) \in \Omega\), \(\alpha(x, y)\), \(\beta(x, y)\) and \(\gamma(x,y)\) are strictly positive scalar functions of the cost-flow relationship reflecting the local characteristics of the road streets, \(f(x, y) = (f_x(x, y), f_y(x, y))\) is a vector that represents the flow state in the city, \(f_x(x, y)\) and \(f_y(x, y)\) are the flow flux in the \(x\) and \(y\) directions respectively, and

\[
|f(x, y)| = \sqrt{f_x(x, y)^2 + f_y(x, y)^2}
\]

(3)

is the norm of the flow vector at \((x,y)\). This cost-flow relationship is an extension to the linear model of Sasaki et al. (14).

Inside the continuous domain of the city \(\Omega\), the flow vector and travel demand must satisfy the flow conservation condition as

\[
\nabla \cdot f(x, y) + q(x, y) = 0, \quad \forall (x, y) \in \Omega.
\]

(4)

Assuming that there is no traffic flow crossing the boundary of the city, we have \(f = 0, \forall (x, y) \in \Gamma\). It is, however, not too difficult to extend the model to deal with the case that \(f \cdot n = g(x, y)\) on the boundary \(\Gamma\), where \(n\) is the normal vector on the boundary and \(g\) is a function representing the given demand distribution entering or leaving the city through the boundary.

Each interchange or CBD is of finite size and is enclosed by a clockwise boundary segment \(\Gamma_n\), \(n = 0, 1, 2, \ldots, N\). Denote \(\Omega_0\) as the catchment of the CBD for the travelers who move directly from their demand locations to the CBD without traveling on the freeway network, and \(\Omega_n\) as the catchment area of interchange \(n\). The travel demand at interchange \(n\) becomes

\[
Q_n = \iint_{\Omega_n} q(x, y) \, d\Omega, \quad n = 0, 1, 2, \ldots, N.
\]

(5)

where \(Q_0\) is the demand for travel directly to the CBD via the continuum only. Denote \(Q = (Q_n, n = 0, 1, 2, \ldots, N)\). From the flow conservation principle, at the CBD and the interchanges we have:

\[
\int_{\Gamma_{in}} f \cdot n \, d\Gamma + Q_n = 0, \quad n = 0, 1, 2, \ldots, N.
\]

(6)
Discrete System

The discrete freeway network is described by a graph \( G(K, A) \), where \( K \) is the set of discrete nodes including the CBD, the interchanges and the intermediate nodes, and \( A \) is the set of links, as shown in Figure 1b. Denote \( v_a \) as the traffic flow on link \( a \) of the freeway network, and the set of link flows as \( \mathbf{v} = (v_a, a \in A) \). The travel cost function on link \( a \) is described by

\[
\bar{\alpha}_a (v_a) = \bar{\alpha}_a + \bar{\beta}_a v_a^\gamma.
\]

where \( \bar{\alpha}_a, \bar{\beta}_a, \) and \( \gamma \) are strictly positive coefficients. Let \( P_n \) be the set of paths from interchange \( n \) to the CBD, and the traffic flow on path \( p \) be \( h_{np} \), \( p \in P_n, n = 1, 2, \ldots, N \). The traffic demand \( Q_n \) that is attracted to interchange \( n \) as determined from equation (6) will form the origin-destination (O-D) demand in the discrete network, and \( Q \) represents the O-D matrix for the network. Denote \( \delta_{npa} \) as the link path incidence value, where 1 when the path \( p \) from interchange \( n \) uses link \( a \), and \( \delta_{npa} = 0 \) otherwise. We have

\[
v_a = \sum_{n=1}^{N} \sum_{p \in P_n} \delta_{npa} h_{np}.
\]

Further denote \( U_n \) as the equilibrium travel cost from interchange \( n \) to the CBD via the freeway network. Naturally, we have \( U_0 = 0 \).

FORMULATION OF THE PROBLEM

Continuous System

The problem of user equilibrium for the continuous system can be formulated as the following mathematical program.

\[
\text{(P1) } \min \int_{\Omega} \Omega \Omega + \gamma = q \Omega, \quad (9a)
\]

subject to

\[
\nabla \cdot \mathbf{f} + q = 0, \quad \forall (x, y) \in \Omega, \quad (9b)
\]

\[
\mathbf{f} = 0, \quad \forall (x, y) \in \Gamma, \quad (9c)
\]

\[
\int_{\Gamma_n} \mathbf{f} \cdot \mathbf{n} \, d\Gamma + Q_n = 0, \quad n = 0, 1, 2, \ldots, N. \quad (9d)
\]

Consider the following Lagrangian,

\[
\Pi = \sum_{n=1}^{N} U_n Q_n + \int_{\Omega} \Omega \left( \alpha | \mathbf{f} | + \frac{\beta}{\gamma + 1} | \mathbf{f} |^{\gamma+1} - \int_0^q D^{-1}(\omega) \, d\omega + u(\nabla \cdot \mathbf{f} + q) \right) \, d\Omega
\]

\[
\quad + \int_{\Gamma} \mathbf{w} \cdot \mathbf{f} \, d\Gamma + \sum_{n=0}^{N} \pi_n \left( \int_{\Gamma_n} \mathbf{f} \cdot \mathbf{n} \, d\Gamma + Q_n \right), \quad (10)
\]

where \( u(x, y), \mathbf{w}(x, y) = (w, x, y), w, (x, y) \), and \( \pi_n \) are the Lagrange multipliers that are associated with the constraints (9b), (9c) and (9d) respectively. From the variational principle, let \( \delta \mathbf{f} = (\delta f_x, \delta f_y) \) be arbitrary functions that vanish on the boundary of the domain, i.e. \( \delta \Omega = \mathbf{0}, \forall (x, y) \in \Gamma \). We can easily show that
\[
\delta \Pi = \int_{\Omega} \left\{ \frac{1}{|f|^2} f \cdot \delta f + \beta |f|^2 \delta f - D^{-1}(q) \delta q + u(\nabla \cdot \delta f + \delta q) + \delta u(\nabla \cdot f + q) \right\} \, d\Omega \\
+ \int_{\Gamma} (w \cdot \delta f + \delta w \cdot f) \, d\Gamma + \sum_{n=0}^{N} U_n \delta Q_n + \sum_{n=0}^{N} \delta \pi_n \left( \int_{\Gamma \cap \Omega} f \cdot n \, d\Gamma + Q_n \right) \\
+ \sum_{n=0}^{N} \pi_n \left( \int_{\Gamma \cap \Omega} \delta f \cdot n \, d\Gamma + \delta Q_n \right).
\]

By substituting \( \nabla \cdot (u \delta f) = \nabla u \cdot \delta f + u \nabla \cdot \delta f \), we have
\[
\delta \Pi = \int_{\Omega} \left\{ \frac{1}{|f|^2} f \cdot \delta f - D^{-1}(q) \delta q + \nabla \cdot (u \delta f) - \nabla u \cdot \delta f + u \delta q + \delta u(\nabla \cdot f + q) \right\} \, d\Omega \\
+ \int_{\Gamma} (w \cdot \delta f + \delta w \cdot f) \, d\Gamma + \sum_{n=0}^{N} U_n \delta Q_n + \sum_{n=0}^{N} \delta \pi_n \left( \int_{\Gamma \cap \Omega} f \cdot n \, d\Gamma + Q_n \right) \\
+ \sum_{n=0}^{N} \pi_n \left( \int_{\Gamma \cap \Omega} \delta f \cdot n \, d\Gamma + \delta Q_n \right) \\
\]

According to Gauss’s integral theorem (divergence theorem),
\[
\int_{\Omega} \nabla \cdot (u \delta f) \, d\Omega = \int_{\Omega} u(\delta f \cdot n) \, d\Gamma + \sum_{n=0}^{N} \int_{\Gamma \cap \Omega} u(\delta f \cdot n) \, d\Gamma,
\]
we can show that
\[
\delta \Pi = \int_{\Omega} \left\{ \frac{1}{|f|^2} f \cdot \delta f - \nabla u \right\} \cdot \delta f \, d\Omega + \int_{\Omega} (u - D^{-1}(q)) \delta q \, d\Omega + \int_{\Omega} \delta u(\nabla \cdot f + q) \, d\Omega \\
+ \int_{\Gamma} \delta w \cdot f \, d\Gamma + \int_{\Gamma} (u n + w) \cdot \delta f \, d\Gamma + \sum_{n=0}^{N} \left( U_n + \pi_n \right) \delta Q_n \\
+ \sum_{n=0}^{N} \delta \pi_n \left( \int_{\Gamma \cap \Omega} f \cdot n \, d\Gamma + Q_n \right) + \sum_{n=0}^{N} \left( u + \pi_n \right) \delta f \cdot n \, d\Gamma.
\]

As \( \delta f, \delta q, \delta u, \delta w, \delta Q_n \), and \( \delta \pi_n \), \( n = 0, 1, 2, \ldots, N \), are arbitrary functions, and \( \delta f \) vanish on the boundary \( \Gamma \), the stationary point of the Lagrangian requires that
\[
\left( \alpha + \beta |f|^2 \right) \left( \frac{f}{|f|} \right) - \nabla u = 0, \quad \forall (x,y) \in \Omega, \quad (14)
\]
\[
u = D^{-1}(q) \quad \text{or} \quad q = D(u), \quad \forall (x,y) \in \Omega, \quad (15)
\]
\[
\nabla \cdot f + q = 0, \quad \forall (x,y) \in \Omega, \quad (16)
\]
\[
f = 0, \quad \forall (x,y) \in \Gamma, \quad (17)
\]
\[
\int_{\Gamma \cap \Omega} f \cdot n \, d\Gamma + Q_n = 0, \quad n = 0, 1, 2, \ldots, N, \quad (18)
\]
\[
U_n + \pi_n = 0, \quad n = 0, 1, 2, \ldots, N, \quad (19)
\]
\[
u + \pi_n = 0, \quad \forall (x,y) \in \Gamma \cap \Omega, \quad (20)
\]

From equations (19) and (20), we have \( \nu = U_n, \forall (x,y) \in \Gamma \cap \Omega \). From equations (15) and (16), the demand function and flow conservation equation are automatically satisfied. Moreover, from equation (14), we can show that the traffic flow vector is parallel to the gradient of the Lagrange multiplier \( \nu \), i.e. \( f // \nabla \nu \) and
\[
\nu = \alpha + \beta |f|^2 = |\nabla \nu| \quad (21)
\]

For each traveler at a location \( H \), the total cost on any used path \( p \) from \( H \) to the CBD via \( O_n \) (for the case of \( n = 0 \), which denotes when the traveler is traveling directly to the CBD in the continuum) can be expressed as
\[ C_{np} = U_n + \int_p c \, ds = U_n + \int_p |\nabla u| \, ds = U_n + \int_p \nabla u \cdot ds = U_n + u(H) - u(O_n) = u(H) \]  

(22)

which is independent of the used paths and the interchange chosen. Therefore, the total costs on all used paths are equal. For any unused path \( \tilde{p} \) from \( H \) to the CBD via \( O_n \), the total cost is

\[ C_{np} = U_n + \int_p c \, ds = U_n + \int_p |\nabla u| \, ds \geq U_n + \int_p \nabla u \cdot ds = U_n + u(H) - u(O_n) = u(H) \]  

(23)

The inequality in the above derivation occurs because in some regions along the path \( \tilde{p} \) the vectors \( \nabla u \) and \( ds \) are not parallel, and \( |\nabla u| ds > \nabla u \cdot ds \) for some segments \( \zeta \in \tilde{p} \). Therefore, for any unused paths the total cost is greater than or equal to that of the used paths. This satisfies the user equilibrium conditions (22) when the elements of \( U = (U_n, n = 0,1,\ldots,N) \) represent the corresponding equilibrium travel costs in the discrete network. The Lagrange multiplier that is associated with the flow conservation constraint can be interpreted as the total cost potential (which is the travel cost to an interchange in the continuum plus the equilibrium travel cost from the interchange to the CBD in the discrete network).

The optimization program (P1) can be written in the following abstract form:

\[ Q = F(U) \]  

(24)

For a given set of the equilibrium travel costs from the interchanges to CBD, the program P1 determines the catchment of each interchange as well as the CBD, which will form the O-D demand in the discrete network.

Discrete System

The user equilibrium problem in the discrete network becomes one of solving the following mathematical program:

\[ \text{(P2)} \quad \text{Minimize} \quad Z(v) = \sum_{a \in A} \int_0^{v_a} c_a(\omega) \, d\omega = \sum_{a \in A} \bar{c}_a v_a + \frac{\bar{\beta}_a}{\gamma + 1} v_a^\gamma + 1 \]  

(25a)

subject to

\[ Q_n = \sum_{p \in P_n} h_{np}, \quad n = 1,2,\ldots,N \]  

(25b)

\[ v_a = \sum_{n=1}^N \sum_{p \in P_n} \delta_{npa} h_{np}, \quad \forall a \in A \]  

(25c)

\[ h_{np} \geq 0, \quad \forall p \in P_n, n = 1,2,\ldots,N \]  

(25d)

The satisfaction of user equilibrium conditions can be found in the work of Sheffi (23). Denote \( \bar{c}_{np} \) as the travel cost on path \( p \) from the interchange location \( n \) to the CBD, and \( \bar{U}_n \) as the corresponding minimum travel cost, i.e. \( \bar{U}_n = \min(\bar{c}_{np}, \forall p \in P_n) \).

The optimization program (P2) can be written in the following abstract form:

\[ \bar{U} = G(Q) \]  

(26)

For a given origin-destination matrix, P2 determines the equilibrium travel costs from the interchange to the CBD.

Fixed point problem

From equations (24) and (26), we have

\[ \bar{U} = G(F(U)) \]  

(27)
If we can find a mutually consistent $U^*$ so that $U^* = G(F(U^*))$, then $U^*$ is said to be a fixed point solution that satisfies all of the functional relationships and user equilibrium conditions of the combined problem.

**SOLUTION ALGORITHM**

**Continuous System**

The whole city region is first discretized into a set of triangular finite elements (with an example as shown in Figure 3). Let $M_N$, $M_B$, and $M_C$ be the number of finite element nodes (FEN) in the whole region, the number of FEN on $\Gamma$, and the total number of FEN on the boundaries of interchanges respectively, and CBD, $\Gamma_{cn}$, $n = 0, 1, 2, \ldots, N$. As the cost potential on the boundary of the interchange $n$ (or CBD) is equal to $U_n$, the optimization program $P_1$ can be solved by determining the stationary point of the following modified Lagrangian,

$$
\bar{\Pi} = -\sum_{n=0}^{N} U_n \int_{\Gamma_{cn}} \mathbf{f}_e \cdot \mathbf{n}_e \, d\Gamma
+ \sum_{e \in \Omega} \int_{\Omega_e} \left( \alpha_e |\mathbf{f}_e| + \frac{\beta_e}{\gamma_e + 1} |\mathbf{f}_e|^{\gamma_e + 1} + u_e (\nabla \cdot \mathbf{f}_e + D(u_e)) - \int_0^{D(u_e)} D^{-1}(\omega) \, d\omega \right) \, d\Omega
+ \sum_{i \in \Gamma_{cn}} (w_{xi} f_{xi} + w_{yi} f_{yi}) + \sum_{n=0}^{N} \sum_{i \in \Gamma_{cn}} \sigma_i (u_i - U_n),
$$

where $\Omega_e$ is the sub-domain of an element $e$, $\alpha_e$, $\beta_e$, and $\gamma_e$ are the coefficients of the cost-flow relationship that are assumed to be constant within the element, $\mathbf{f}_e = (f_x(x,y), f_y(x,y))$ and $u_e(x,y)$ are the flow vector and cost potential respectively, and $\mathbf{n}_e$ is the normal vector along the edge between two adjacent nodal points on $\Gamma_{cn}$. The summation in the third term consists of all of the nodal points on the boundary $\Gamma$. It will be shown later that the flows are approximated by linear functions within an element, and hence setting the flows at any two adjacent nodal points zero is sufficient to ensure that the flows on the edge between the two nodes vanish. For the last term, the summation comprises all of the nodal points on the boundary $\Gamma_{cn}$, and $\sigma_i$, $i = 1, 2, \ldots, M_C$ are the Lagrange multipliers that are associated with such nodal points. Again, as the cost potential within an element is approximated by a linear function, the cost potential will be equal to $U_n$ on the corresponding boundary of the interchanges and the CBD.

Inside a finite element, the flow vector and cost potential are approximated by linear functions. With these approximations, the Lagrangian can now be expressed as a function of all of the nodal parameters, i.e. $\bar{\Pi}(\Psi, U^*)$, where $\Psi = \text{Col}(\mathbf{f}, \mathbf{u}, \mathbf{w}, \sigma)$, $\mathbf{f} = \text{Col}(f_{xi}, f_{yi}, i = 1,2,\ldots,M_N)$, $\mathbf{u} = \text{Col}(u_i, i = 1,2,\ldots,M_N)$, $\mathbf{w} = \text{Col}(w_{xi}, w_{yi}, i = 1,2,\ldots,M_B)$ and $\sigma = \text{Col}(\sigma_i, i = 1,2,\ldots,M_C)$ are the control variables, for a given $U^*$. The stationary point of the Lagrangian, $\Psi^*$, can be located by setting $\mathbf{R}(\Psi^*, U^*) = \nabla_{\Psi} \bar{\Pi}(\Psi^*, U^*) = 0$ for a given $U^*$ by using a Newtonian algorithm. Details of the finite element procedure for this problem can be found in the work of Yang and Wong (20). From the implicit function theorem, the sensitivity of solution $\Psi^*$ with respect to the perturbed parameters $U$ at $U^*$ can be expressed as
\[ \nabla_{U} \Psi^{*} = -\left[ \nabla_{\Psi} R(\Psi^{*}, U^{*}) \right]^{-1} \nabla_{U} R(\Psi^{*}, U^{*}) \]
\[ = -\left[ \nabla_{\Psi}^{2} \Pi(\Psi^{*}, U^{*}) \right]^{-1} \nabla_{U} \Psi \Pi(\Psi^{*}, U^{*}), \]

where \( \nabla_{\Psi}^{2} \Pi(\Psi^{*}, U^{*}) \) and \( \nabla_{U} \Psi \Pi(\Psi^{*}, U^{*}) \) can be explicitly obtained by differentiating equation (28). Then, by substituting the results that are obtained from equation (29) into equation (7), we can determine the sensitivity of the O-D matrix \( Q \) in the discrete network with respect to the equilibrium travel costs at the interchanges \( U \), i.e.
\[ \nabla_{U} Q = \nabla_{U} F(U). \] (30)

### Discrete System

The optimization program \( P2 \) can be solved by means of the Frank-Wolfe solution algorithm (23), and the sensitivity of the O-D travel costs \( \bar{U} \) with respect to the O-D demand \( Q \) can then be obtained as in the work of Tobin and Friesz (24), i.e.
\[ \nabla_{Q} \bar{U} = \nabla_{Q} G(Q). \] (31)

### Newtonian Algorithm

The fixed point problem in equation (27) can now be solved by a Newtonian algorithm as follows. The problem is to find a \( U \) so that
\[ E(U) = U - G(F(U)) = 0, \] (32)
where \( E(U) \) is a column vector that measures the discrepancies in O-D travel costs at the interchanges from the two optimization sub-programs for the continuous and discrete systems. Let \( U^{*} \) be an approximate solution to the problem. Expanding \( E(U) \) by Taylor’s series around the point \( U^{*} \), we have
\[ E(U) \equiv E(U^{*}) + \left[ \nabla_{U} E(U^{*}) \right] (U - U^{*}), \] (33)
where \( \nabla_{U} E(U^{*}) \) is the Jacobian matrix as evaluated at \( U^{*} \). For a stationary point, the derivatives of \( E(U) \) with respect to all of the variables vanish. Therefore, a better solution can be obtained by
\[ U = U^{*} - \left[ \nabla_{U} E(U^{*}) \right]^{-1} E(U^{*}), \] (34)
where \( E(U^{*}) \) is evaluated by equation (32), which requires the solutions of the two optimization sub-problems for the continuous and discrete systems, and the Jacobian matrix is obtained from the sensitivity analyses of the two sub-problems,
\[ \nabla_{U} E(U^{*}) = I - \left[ \nabla_{Q} G(Q) \right] \nabla_{U} F(U^{*}). \] (35)

### Solution Procedure

The solution procedure is summarized as follows.

**Step 1:** Set \( k = 0 \). Set the travel costs on all links as the free-flowing costs \( (\bar{c}_{a}, \forall a \in A) \) and find the shortest paths from all interchanges to the CBD. Set the minimum path costs as \( U^{(k)} \).
Step 2: Solve optimization sub-problem P1 for the continuous system to obtain the O-D demand \( Q^{(k)} = F(U^{(k)}) \). Conduct sensitivity analysis to determine \( \nabla_U F(U^{(k)}) \).

Step 3: Solve optimization sub-problem P2 for the discrete system to obtain the minimum equilibrium path costs \( U^{(k)} = G(Q^{(k)}) \). Conduct sensitivity analysis to determine \( \nabla_Q G(Q^{(k)}) \).

Step 4: Evaluate \( E^{(k)} = U^{(k)} - \bar{U}^{(k)} \) and \( \nabla_U E(U^{(k)}) = I - \left[ \nabla_Q G(Q^{(k)}) \right] \nabla_U F(U^{(k)}) \).

Step 5: If \( |E^{(k)}| < \varepsilon \), an acceptable error, then stop and \( U^{(k)} \) is the solution.

Step 6: Otherwise, find \( U^{(k+1)} = U^{(k)} - \left[ \nabla_U E(U^{(k)}) \right]^{-1} E^{(k)} \).

Step 7: Set \( k = k + 1 \) and go to Step 2.

EXAMPLE

Consider the modeled city, which consists of 4 interchanges and 6 discrete links, as shown in Figure 2. The CBD is located in the southwest of the city. The travel demand function is

\[
q = 100 \exp(-0.5u)
\]  
(36)
everywhere in the city, where \( q \) and \( u \) are measured in veh/h/km² and h respectively. The cost-flow relationship for the continuous system is specified as

\[
c = 0.01 + 0.1 \times 10^{-4} \|f\|^{1.2}
\]  
(37)
throughout the city, where \( c \) and \( \|f\| \) are measured in h/km and veh/h/km respectively. The cost-flow relationship on links for the discrete system is specified as

\[
\bar{c}_a = \bar{c}_{a0} + 2.62 \left( \frac{v_a}{C_a} \right)^5
\]  
(38)
where the parameters \( \bar{c}_{a0} \) and \( C_a \) are listed in Table 1, and \( \bar{c}_a \) and \( v_a \) are measured in h and veh/h respectively. The finite element discretization of the continuous system is shown in Figure 3. The maximum acceptable error \( \varepsilon \) for the solution algorithm is set as 0.1%.

The convergence characteristics of the solution algorithm are shown in Figure 4. The solution converges quickly in 4 iterations. The results of sensitivity analyses provide a good indication of the search direction that leads to the fixed point solution. The flow vectors in the city are plotted in Figure 5, from which the path from a home location to the CBD or interchange can be traced by locating the streamline. The catchment area that is captured by the CBD or interchange can also be observed from the directions that the travelers are moving. The traffic flows in the continuum and discrete network of the city are shown in Figure 6. The traffic flows in the discrete network is also tabulated in the last two columns of Table 1. Note that Link 4 in the discrete network is unused, whereas there are two used paths from origin O4 to the CBD via Links 5 and 6 respectively. This case cannot be solved by the original Yang et al. (21) algorithm. Serious traffic congestion is observed at the CBD and at all of the interchanges. Figures 7 and 8 show respectively the contour plots of the total travel cost and travel demand from home locations to the CBD. The travel demand generally decreases with the distance from the CBD. However, it is interesting to observe that the travel demand increases locally when approaching any of the interchanges, to take advantage of the high speed movements on the discrete network. This also reflects the realistic situation that high development potential is usually concentrated at major highway interchanges. For this example, the total demand that is generated is 31,018 veh/h. The total travel demands
that are attracted to interchanges $O_1$, $O_2$, $O_3$ and $O_4$ are 5,600, 6,331, 5,536, and 6,269 veh/h respectively. The travel demand for traveling directly to the CBD without boarding the freeway network is 7,282 veh/h.

CONCLUSION

In this paper, we have developed a traffic equilibrium model for the discrete/continuous transportation system, in which a city with a highly compact central business district (CBD) is considered. In the city, the commuters’ origins are continuously dispersed and the travel demand is dependent on the total travel cost to the CBD. The transportation system has been divided into two layers: major freeways and dense surface streets. Whereas the major freeway network has been modeled with the conventional discrete network approach, the dense surface streets have been approximated as a continuum. Travelers to the CBD can either travel in the continuum (surface streets) and then exchange to the discrete network (freeways) at an interchange (ramp) before moving to the CBD on the discrete network, or travel directly to the CBD in the continuum. Specific travel cost-flow relationships for the two layers of transportation facilities have been considered. In the model, the commuters follow a user equilibrium route choice pattern, in which for a particular origin no traveler can reduce their individual travel cost to the CBD by unilaterally changing routes.

The problem has been formulated as a simultaneous optimization program with two sub-problems. One sub-problem is a traffic assignment problem from the interchanges to the CBD in the discrete network, and the other is a traffic assignment problem with multiple centers (i.e. the interchange points and the CBD) in the continuous system. This simultaneous optimization program has been shown to belong to the class of fixed point problems. Based on the sensitivity analyses of the two sub-problems, a Newtonian algorithm has been proposed to solve the resultant simultaneous optimization program. The advantages of the formulation and solution algorithm are that we can make use of the state-of-the-art developments in continuum and discrete network modeling techniques and recent advances in sensitivity analyses in traffic modeling. A numerical example has been given to demonstrate the effectiveness of the proposed methodology.

The above methodology can be easily extended to the case of multi-commodity (or multi-destination), in which the multi-commodity continuum model in Wong (16) is combined with the network equilibrium model with multiple origins and destinations. So far, the isotropic cost function used in the continuum is rather restrictive. Research has been undertaken to relax this assumption, which opens the way to analyze a system with anisotropic properties caused by roadway configurations on the surface streets. The calibration and validation of the proposed methodology in the real world is also an interesting research topic to pursue.

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REFERENCES


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FIGURE 6 The traffic flows in the continuum and freeway network
FIGURE 7 The contour plot of the total travel cost in the city
FIGURE 8 The contour plot of the travel demand in the city
## TABLE 1 The input and output data of the discrete network.

<table>
<thead>
<tr>
<th>Link</th>
<th>Length (km)</th>
<th>$C_a$ (veh/h)</th>
<th>$\bar{v}_{a0}$ (h)</th>
<th>$\nu_a$ (veh/h)</th>
<th>$c_a$ (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.6</td>
<td>6,000</td>
<td>0.108</td>
<td>5,600</td>
<td>0.309</td>
</tr>
<tr>
<td>2</td>
<td>7.8</td>
<td>10,000</td>
<td>0.098</td>
<td>9,305</td>
<td>0.278</td>
</tr>
<tr>
<td>3</td>
<td>15.0</td>
<td>12,000</td>
<td>0.188</td>
<td>8,831</td>
<td>0.295</td>
</tr>
<tr>
<td>4</td>
<td>25.0</td>
<td>4,000</td>
<td>0.313</td>
<td>0</td>
<td>0.313</td>
</tr>
<tr>
<td>5</td>
<td>13.6</td>
<td>8,000</td>
<td>0.170</td>
<td>2,974</td>
<td>0.173</td>
</tr>
<tr>
<td>6</td>
<td>12.2</td>
<td>8,000</td>
<td>0.152</td>
<td>3,295</td>
<td>0.156</td>
</tr>
</tbody>
</table>
FIGURE 1 The discrete/continuous transportation system.

(a) Combined System

(b) Discrete System

(c) Continuous System

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Major freeway link

Central business district

Major interchange
FIGURE 2  The example city.
FIGURE 3 The finite element discretization of the continuum of the city.
FIGURE 4 The convergence characteristics of the solution algorithm.
FIGURE 5 The vector plot of traffic movements in the continuum of the city.
FIGURE 6 The traffic flows in the continuum and freeway network.
FIGURE 7 The contour plot of the total travel cost in the city.

Unit: minutes
FIGURE 8 The contour plot of the travel demand in the city.